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THE

MATHEMATICAL GAZETTE.

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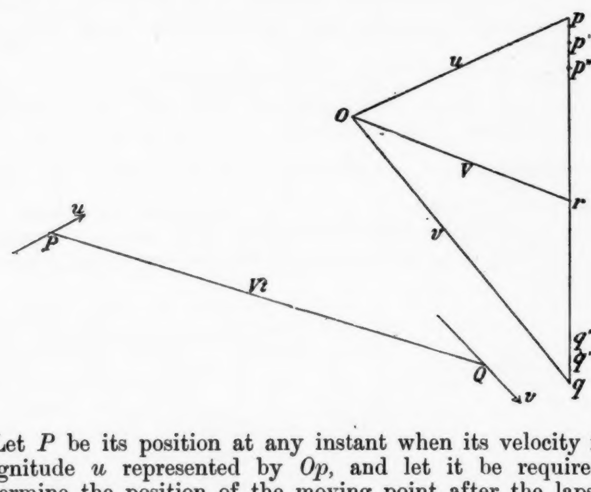
A CHAPTER IN ELEMENTARY DYNAMICS.

THE following is suggested as a geometrical treatment of uniformly accelerated motion.

Suppose that a moving point possesses an acceleration of constant magnitude f in a constant direction.



Let P be its position at any instant when its velocity is of magnitude u represented by Op, and let it be required to determine the position of the moving point after the lapse of time t.

Draw pq of length ft units in the direction of the acceleration. Then Oq represents the velocity v of the moving point after time t.

Bisect pq at r. Then Or represents the velocity V after time $\frac{t}{2}$.

Let the time t be divided into n equal intervals τ of time, where n is an integer; and divide pq into n equal portions in the points $p', p'', \ldots q'', q'$.

Now imagine the moving point to move during each interval τ with the velocity it has at the beginning of that interval. Then it would have displacements due to velocities represented by Op, Op', Op'', ... Oq'', Oq', each occurring for time τ.

Now the displacement due to a velocity represented by Op occurring for time τ is equivalent to two displacements due to velocities represented by Or and rp, each occurring for the same time τ.

Also each of the other displacements can be similarly treated.

Hence the whole displacement is equivalent to n displacements due to velocity V, each occurring for time τ, together with displacements due to velocities represented by rp, rp', rp'', ... rq'', rq', each occurring for time τ.

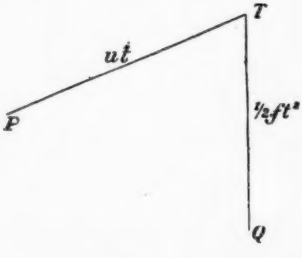
But these latter displacements, with the exception of the first of them, cancel one another.

Hence the whole displacement is equivalent to a displacement due to velocity V occurring for time $n\tau$, together with a displacement due to velocity represented by rp occurring for time τ.

Thus the whole displacement is equivalent to a displacement due to velocity V occurring for time t, together with a displacement $-\frac{ft^2}{2n}$ in the direction of the acceleration.

Now when n is increased without limit, the assumed motion ultimately agrees with that which actually takes place. Also the displacement $-\frac{ft^2}{2n}$ ultimately vanishes. Hence the actual displacement is that due to velocity V occurring for time t.

Thus, draw PQ in the direction of Or, and make it Vt units of length. Then Q is the position of the moving point at the end of time t.



It will be seen that the principle involved in the above method of proof is the *Independence of Displacements*, namely:

If a point has a number of displacements, occurring either simultaneously or successively, they may be taken successively in any order.

COR. I. The velocity V is equivalent to two velocities represented by Op and pr.

Hence the displacement PQ is equivalent to two displacements ut and $\frac{1}{2}ft^2$ in the directions of u and f respectively.

Thus the position of Q can be determined by drawing PT of length ut in the direction of the velocity u, and TQ of length $\frac{1}{2}ft^2$ in the direction of the acceleration f.

If we imagine t to vary, we have at once the character of the path of Q.

In the case of rectilinear motion, where f is in the same direction as u, if s is the measure of the displacement in time t, we have

$$s = ut + \tfrac{1}{2}ft^2.$$

COR. II. Similarly the displacement PQ is equivalent to two displacements vt and $-\frac{1}{2}ft^2$ in the directions of the velocity v and the acceleration f respectively.

In the case of rectilinear motion, this gives

$$s = vt - \tfrac{1}{2}ft^2.$$

COR. III. The velocity V is half the resultant of velocities represented by Op and Oq.

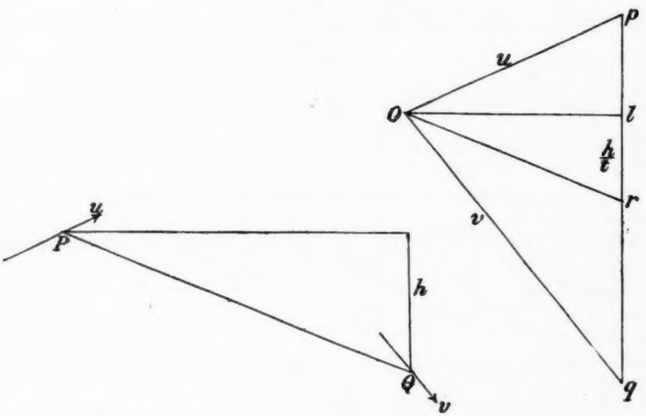
Hence the displacement PQ is equivalent to two displacements $\frac{ut}{2}$ and $\frac{vt}{2}$ in the directions of the velocities u and v respectively.

This shows that the tangent at Q bisects PT.

Also, in the case of rectilinear motion, we have

$$s = \frac{u+v}{2} \cdot t.$$

COR. IV. Draw Ol perpendicular to pq.



Then, if h is the resolved part of the displacement PQ estimated in the direction of the acceleration f, lr must be of length $\frac{h}{t}$.

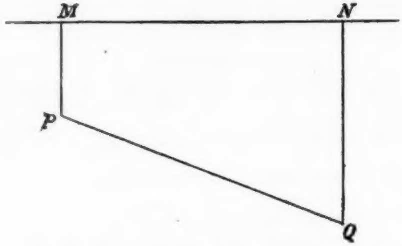
But, by elementary geometry, the difference between the squares on Oq and Op is equal to twice the rectangle contained by lr and pq;

$$\therefore \quad v^2-u^2=2ft\,.\,\frac{h}{t}=2fh.$$

In the case of rectilinear motion, h becomes equal to s, so that we have

$$v^2=u^2+2fs.$$

COR. V. The result of the preceding corollary can be put into a slightly different form.



Draw PM vertically upwards of length $\frac{u^2}{2f}$, and let the horizontal through M meet the vertical through Q in N.

Then

$$\begin{aligned} QN &= PM+h \\ &= \frac{u^2}{2f}+h \\ &= \frac{v^2}{2f}. \end{aligned}$$

Hence, using the equation for rectilinear motion obtained in Cor. IV., we see that, if a point were to start from rest at N and move along NQ with uniform acceleration f, it would reach Q with velocity v.

COR. VI. Draw MS and NS perpendicular to Op and Oq respectively, meeting in S, and SQ' parallel to Oq to meet NQ produced in Q'.

Let u_1 be the measure of Ol. Then, as $PQ=Vt$, it follows that $MN=u_1t$.

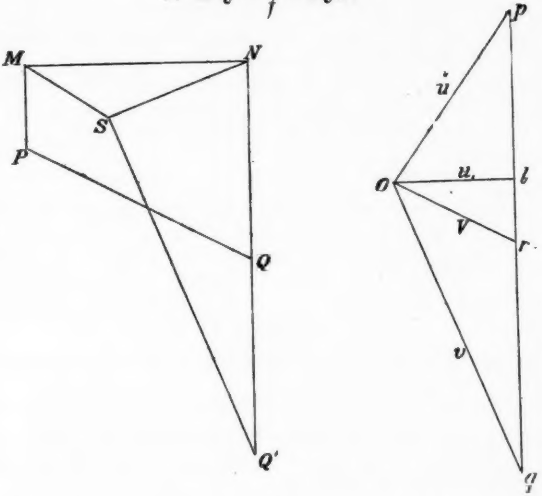
Now the triangles MSN and pOq are similar, the sides of MSN being $\frac{u_1}{f}$ times those of pOq.

Hence, for different values of t, MS is of constant length $\frac{uu_1}{f}$ drawn from a fixed point M in a fixed direction.

$\therefore$ S is a fixed point.

Now the triangles SNQ' and lOq are similar, and $SN=\frac{u_1 v}{f}$, so that the sides of SNQ' are $\frac{v}{f}$ times those of lOq.

$$\therefore\ NQ'=\frac{v^2}{f}=2QN.$$



$\therefore$ Q is the middle point of the hypotenuse of the right-angled triangle SNQ'. $\qquad \therefore\ SQ=QN.$

Thus, for different values of t, the path of Q is such that its distance from the fixed point S is equal to its perpendicular distance from the fixed straight line MN. W. J. DOBBS.

SOME CURIOSITIES IN DIVISION.

WHILE engaged a few years ago in the consideration of various devices for the abbreviation of arithmetical operations, I found an article "On the Operations of Arithmetic," in the eighth edition of the *Encyclopaedia Britannica* (1853), containing among other matter a description of a short method for dividing a number by 99, 999, 9999, etc. The article in question does not appear to be due to Leslie as I at first supposed though it is joined to one undoubtedly by him "On the History of Arithmetic." It seems little more than a reprint with slight alterations of an article in previous editions of the *Encyclopaedia Britannica.* The following is quoted from the third edition (1797), which is the earliest to which I have access:

"To divide by 9, 99, 999 or any number of 9's transcribe under the dividend part of the same, shifting the highest figure as many places to the right hand as there are 9's in the divisor. Transcribe it again with the like change of place as often as the length of the dividend admits; add these together and cut off as many figures from the right as there are 9's in the divisor. The figures which remain on the left-hand side compose the quotient and those cut off the remainder.

If there be any carriage to the unit place of the quotient, add the number carried likewise to the remainder as in Ex. 2, and if the figures cut off be all 9's add 1 to the quotient and there is no remainder.

Examples.

```
1st.]  99 ) 324123      2nd.]  99 ) 547825      3rd.]  999 ) 476523
              3241                    5478                     476
                32                      54                 -------
           -------                 -------                 476|999
           3273|96                 5533|57                       1
  Q.=3273, R.=96.                        1            Q.=477, R.=0."
                             Q.=5533, R.=58
```

These instructions are followed by a clear demonstration of the validity of the process.

Of course, the method would apply to 9 itself, but without contraction it would be more tedious than ordinary division. But the common method for 'casting out nines' supplies us with a convenient contraction.

Example. To divide 6739163 by 9.

```
6739163
-------
636562
112233...8
----------
748795...8
```

Here in 'casting out nines' we obtain in succession 6, 13, 16, 25, 26, 32 (which we write diagonally upwards), and finally 35 which is 3 (for the units' place), and 8 over. Then, by addition, we get 748795 for the quotient.

It will be noticed that we have here merely abbreviated the process described in the *Encyclopaedia Britannica*, and that if we followed out the instructions in full the work would stand thus:

```
6739163
----------
673916...3
 67391...6
  6739...1
   673...9
    67...3
     6...7
         6
----------
748795...8
```

A little consideration will show that the method applies to 101, 1001, etc., as well as to 99, 999, etc., if we affect the alternate lines obtained by shifting with the minus sign and add algebraically.

The work for dividing 6739163 by 101, 1001 is given below.

$$\begin{array}{r}
\underline{6739163} \\
67391...63 \\
\bar{6}\bar{7}\bar{3}...\bar{9}\bar{1} \\
6...73 \\
\underline{\bar{6}} \\
67\bar{3}24...4\bar{1}
\end{array}
\qquad\qquad
\begin{array}{rr}
\underline{6739163} & \\
6739...163 & \\
\bar{6}...\bar{7}\bar{3}\bar{9} & \\
 & \underline{6} \\
6733 & \bar{6}30 \\
\underline{\bar{1}} & \underline{101} \\
6732 & 431
\end{array}$$

Q. = 66724, *R.* = 39.

It applies to 11 itself but as in the case of 9 it would, unless contracted, be more tedious than actual division. But De Morgan's method for 'casting out elevens' supplies a convenient contraction.

Example. To divide 6739163 by 11.

$$\begin{array}{rr}
\underline{6739163} & \\
6127\bar{6}2 & \bar{9} \\
\underline{1\bar{1}} & 2 \\
6127\bar{5}1 & 2
\end{array}$$

Here De Morgan's method has been used for 'casting out elevens'; 6 from 7, 1 ; 1 from 3, 2 ; 2 from 9, 7 ; 7 from 1, $\bar{6}$; $\bar{6}$ from 6, 12 ; and finally 12 from 3, $\bar{9}$. To get *R.* which must be positive we say 11 into 9, $\bar{1}$ (for the units' place) and 2 over.

To get the quotient we add algebraically, and write 612651, the 65 replacing $7\bar{5}$, *i.e.* $70-5$.

Comparing this with the example on division by 9, it will be found to be an abbreviation of a process which is exactly that of the *Encyclopaedia Britannica*, except that the alternate numbers to be added are affected with the minus sign.

The extension of the methods for

$$10^n \pm 1 \text{ to } 10^n \pm 2,\ 10^n \pm 3,\ 10^n \pm 4, \text{ etc.},$$

is pretty obvious; the reasoning supplied in the *Encyclopaedia Britannica* showing that we have to multiply by 2, 3, 4, etc. at the same time that we shift.

Example. To divide 6739163 (i.) by 98 ; (ii.) by 102.

(i.)

$$\begin{array}{rr}
\underline{6739163} \\
67391...63 \\
1347...82 \\
26...94 \\
52 \\
\overline{68766} \quad \overline{91} \\
\underline{4} \\
95
\end{array}$$

(ii.)

$$\begin{array}{rr}
\underline{6739163} & \\
67391 & 63 \\
\bar{1}\bar{3}\bar{4}\bar{7} & \bar{8}\bar{2} \\
26 & 94 \\
 & \underline{\bar{5}\bar{2}} \\
\overline{66070} & 23
\end{array}$$

Example. To divide 6739163 (i.) by 97 ; (ii.) by 103.

$$\text{(i.)}\quad \begin{array}{rr} \underline{6739163} & \\ 67391\ldots & 63 \\ 2021\ldots & 73 \\ 60\ldots & 63 \\ 1\ldots & 80 \\ & \underline{3} \\ \overline{69475} & 82 \\ & \underline{6} \\ & 88 \end{array} \qquad\qquad \text{(ii.)}\quad \begin{array}{rr} \underline{6739163} & \\ 67391\ldots & 63 \\ \bar{2}\bar{0}\bar{2}\bar{1}\ldots & \bar{7}\bar{3} \\ 60\ldots & 63 \\ \bar{1} & \bar{8}\bar{0} \\ & \underline{3} \\ \overline{6543\bar{1}} & \overline{\bar{3}6} \\ \bar{1} & 103 \\ \overline{65428} & \overline{79} \end{array}$$

As pointed out in *Computation*, p. 71, the method depends on the algebraical identity

$$\frac{1}{x-a}=\frac{1}{x}+\frac{a}{x^2}+\frac{a^2}{x^3}+\frac{a^3}{x^4}, \text{ etc.}$$

This is chiefly useful for numerical computation

(i.) when, as in the examples given, x is some power of 10 and a is not greater than 12.

(ii.) when $a=1$ and x is of the form $p\,.\,10^n$ where p is not greater than 12.

Example.

$$\frac{1}{19}=\frac{1}{20}+\frac{1}{(20)^2}+\frac{1}{(20)^3}+\text{etc.}$$

$$\frac{1}{39}=\frac{1}{40}+\frac{1}{(40)^2}+\frac{1}{(40)^3}+\text{etc.}$$

$$\frac{1}{41}=\frac{1}{40}-\frac{1}{(40)^2}+\frac{1}{(40)^3}-\text{etc.}$$

The application of these is sometimes more effective than would at first sight appear.

Take the well-known case of the recurring decimal equivalent to $\frac{1}{19}$.

$$\frac{1}{19}=\cdot05\left\{1+\frac{1}{20}+\frac{1}{(20)^2}\right\}$$
$$=\cdot\dot{0}5263157894736842\dot{1}.$$

2 into 5, 2 (which we write down after the 5), and 1 over.
2 into 12, 6 (which we write down after the 2).
2 into 6, 3 (which we write down after the 6),
and so on.

Fractions which do not rank under either of these two classes may frequently be reduced to one of them.

It is intended to give examples of a few devices for effecting this reduction in a subsequent paper. EDWARD M. LANGLEY.

MATHEMATICAL NOTES.

62. *Note on Mr. Billups' article "On the Connection between the Inscribed and Escribed Circles of a Triangle"* (p. 177).

We have received two or three communications pointing out that Mr. Billups has been anticipated from more than one source in the results given in his paper. Mr. Billups' paper is, in effect, a partial exposition of what Mr. E. Lemoine has called *la transformation continue* of the triangle.

The central idea of the transformation is, we believe, as follows: Suppose that the base BC of a triangle ABC remains fixed while the vertex A moves away to infinity on the one side of the base BC and returns again into the finite region on the opposite side of the base. The original triangle ABC is thus transformed into a new triangle ABC; and the question arises as to what any magnitude, connected with the original triangle, becomes transformed into in the new triangle. The magnitudes a, b, c become transformed to $a, -b, -c$; A, B, C to $-A, \pi-B, \pi-C$; s, s_a, s_b, s_c to $-s_a, -s, s_c, s_b$; r, r_a, r_b, r_c to $r_a, r, -r_c, -r_b$; R, Δ to $-R, -\Delta$, etc., etc. Besides these Mr. Lemoine considers the effect of the transformation on the coordinates of a point whether trilinear, areal, or cartesian, the altitudes of the triangle, the bisectors of the angles, and other magnitudes. A formula connecting any of the above magnitudes thus gives rise to a new formula, any such being but a different expression of the truth contained in the original.

Dr. J. S. Mackay, in an article on "The Radii of the In-circle and Ex-circles of a Triangle" (*Proceedings of the Edinburgh Mathematical Society*, Vol. XII., 1893-94) gives the following references:

"The greater part of this table (embodying the results mentioned above) is given in the *Lady's and Gentleman's Diary* for 1871, p. 93, and it is due either to the editor of the *Diary*, W. S. B. Woolhouse, or to one of his correspondents, W. B. G. (William Bywater Grove ?). No demonstration, however, is offered of the law of transformation thus enunciated."

"A discussion of this law by Mr. E. Lemoine will be found in the *Bulletin de la Société Mathématique de France*, XIX., 133-141 (1891); in Mr. De Longchamps' *Journal de Mathématiques Elémentaires*, 4th Series, I. 62-69, 91-93, 103-106 (1892); and in *Mathesis*, 2nd Series, II. 58-64, 81-92 (1892). Two articles on the same subject by Edouard Lucas will be found in *Nouvelle Correspondance Mathématique*, II., 384-391 (1876); III., 1-5 (1877)."

In Vol. XIII. of the *Proceedings*, E.M.S. (1894-95), Mr. Lemoine contributes an article ("Étude sur le Triangle et sur certains points de Géométographie") in which the transformation is extended to the tetrahedron; but he does not there give a proof of the method. In the same volume Mr. R. F. Muirhead shows that the transformation obtained by passing from a spherical triangle to its colunar triangle is closely analogous to the one in question, and refers to Casey's *Spherical Trigonometry* for a list of formulae in that connection.

For a proof of the *transformation continue*, given by Lemoine and ascribed to Laisant, we have been referred to the *Report* (1891, Vol. II.) of the French Association for the Advancement of Science.

We are indebted to Dr. Mackay and Mr. Muirhead for the particulars contained in this note.

63. *A Theorem of Infinitesimals, applied to Conics.*

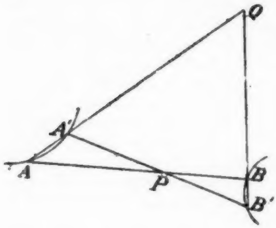
(1) If AB, $A'B'$ are two chords of any curve of finite curvature intersecting at P; and AA', BB' meet at Q; then, as $A'B'$ approaches coincidence with AB, the ratio $AP:PB$ is the limit of $AA'/AQ : BB'/QB$.

For $\qquad AP : PB = AA' \sin AA'P : BB' \sin BB'P$;

and $\qquad \sin AA'P : \sin BB'P = QB' : QA' = QB : QA$

in the limit, when AQ, QB become tangents at A and B. The same result follows by regarding $A'PB'$ as a transversal of the $\triangle AQB$.

To include the case in which P lies in AB produced, it is only necessary to pay attention to the signs of the lengths as indicated by the order of the letters.



(2) To apply this result to a *central* conic we observe that tangents QA, QB subtend at the centre O, triangles of equal area; so that

$$\begin{aligned} AA'/AQ : BB'/QB &= \triangle AOA'/\triangle AOQ : \triangle BOB'/\triangle BOQ \\ &= \triangle AOA' : \triangle BOB'. \end{aligned}$$

Thus $AP : PB = \lim\,(p\delta s : p'\delta s')$, where p, p' are the central perpendiculars on the tangents at A and B, δs, $\delta s'$ the elements of arc intercepted by a consecutive chord through P.

(*a*) *Hence the ratio in which any chord of a central conic is divided by a consecutive chord equals the ratio of the elementary areas subtended at the centre of the curve by the elements of intercepted arc.*

The result in (1) can also be easily applied to the parabola; for, in this case, the projections of AQ, QB on the directrix are equal, so that the ratio $AA'/AQ : BB'/QB$ is equal to the ratio of the projections of AA', BB' on the directrix. Hence in place of (*a*) we have

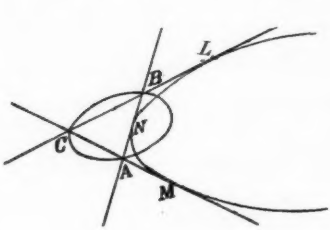
(*b*) *Two consecutive chords of a parabola divide one another in the ratio of the projections of the elements of arc on the tangent at the vertex, or, in other words, in the ratio of the infinitesimal differences in the ordinates at their extremities.*

(3) The theorems (*a*) (*b*) may be applied to establish a theorem due to Mr. R. Pendlebury (Taylor's *Geometry of Conics*, p. 361):

If a triangle be circuminscribed to a parabola and an ellipse, the sum of the eccentric angles of its vertices is constant.

It is known that when two conics are such that a triangle can be inscribed to one and circumscribed to the other, an infinite number can be so circuminscribed (Salmon, § 376).

Suppose α, β, γ the eccentric angles of the vertices of a triangle inscribed to an ellipse and circumscribed to a fixed parabola, L, M, N the points of contact, in symmetrical order.



By 2 (*a*)

$$\frac{d\alpha}{d\gamma} = -\frac{MA}{MC}\,;\ \frac{d\beta}{d\gamma} = -\frac{LB}{LC}\,;$$

and remembering that, by a property of a parabola,

$$LB/LC = AC/MC$$

in magnitude and sign, we obtain

$$\frac{d\alpha}{d\gamma} + \frac{d\beta}{d\gamma} = -\frac{MA+AC}{MC} = -1,$$

or $\qquad \alpha+\beta+\gamma = \text{constant}.$ [Q.E.D.

Replacing the ellipse by another parabola, and applying 2 (*b*), we see that the sum of the ordinates of A, B, C is constant.

Lastly, replacing the ellipse by a hyperbola, we find that the sum of the sectorial areas of the hyperbola, bounded by the central radii OA, OB, OC

and any fixed radius, from which they are reckoned in the same sense, is constant.

I am not aware that this extension has been noticed. It is, however, known that if the coordinates of a point on an hyperbola are expressed by

$$x=a\cosh u,\ \ y=b\sinh u,$$

then $\frac{1}{2}abu$ represents the sectorial area bounded by the radius vector to (x, y) and that to $(a, 0)$. This sectorial area is therefore analogous to the eccentric angle of (x, y) on an ellipse (Greenhill, *Differential and Integral Calculus*, § 35). C. E. M'VICKER.

64. Some other interesting properties of a system of triangles circuminscribed to a parabola and conic may be added to the above.

The question referred to by Mr. M'Vicker in the *Geometry of Conics*, besides stating that in the case of the ellipse $\alpha+\beta+\gamma$ is constant, adds that *the circumcircles of the triangle ABC cut the ellipse in the same point, and have a common radical axis.* This easily follows from what has been already proved. If δ is the eccentric angle of the fourth point D in which the circle ABC cuts the ellipse, we know that $\alpha+\beta+\gamma+\delta$ is an even multiple of two right angles; hence δ is constant, and the point D is fixed. Also the circle ABC passes through another fixed point besides D, viz. the focus S of the parabola. The circles ABC have therefore a common radical axis DS.

The locus of the circumcentres of the triangle ABC is a straight line, bisecting DS at right angles.

Also the locus of the orthocentres is a straight line, the directrix of the parabola.

Again, the locus of the centroids is a straight line.

This last property is a known theorem (R. A. Roberts, *Educational Times Reprint*, Vol. XXIX.). It may be proved by projecting the ellipse orthogonally into a circle. The parabola then projects into another parabola. In the projection the triangles are circuminscribed to a parabola and circle; the circumcentre O is fixed; the orthocentre P runs on the directrix of the parabola; therefore the centroid G runs on a straight line parallel to the directrix, since G lies on OP and divides it in the ratio of 1 to 2. Hence also, in the original figure, the locus of the centroids of the triangles is a straight line.

The property of the orthocentres remains true when the ellipse is replaced by any conic; and we may assume, by the principle of continuity, that the like properties of the circumcentres and centroids also remain true.

The three loci are not in general concurrent; for, if they were, we could place the orthocentre at the point of concurrence, and the circumcentre and centroid would then lie at the same point. The corresponding circuminscribed triangle would then be equilateral; but such a triangle does not in general exist.

When the conic is neither a circle (circumcentre fixed) nor a rectangular hyperbola (orthocentre fixed), and when there is no equilateral triangle circuminscribed to the parabola and conic, the three loci form a triangle. Any set of three points O, G, P lie on a transversal of this triangle; while the ratio $OG : GP$ is constant. Hence the envelope of OP is a parabola which has the three loci for tangents. Hence also the locus of a point which divides OP in any fixed ratio is a straight line touching this parabola. Thus *the locus of the nine-point centres of the circuminscribed triangles is also a straight line.*

The condition that a set of circuminscribed triangles should exist for a parabola and hyperbola is that the tangents to the parabola which are parallel to the asymptotes of the hyperbola should intersect one another on the hyperbola; for, when this is the case, the two tangents with the line at infinity form a circuminscribed triangle. If the two tangents are taken as

axes of coordinates, the equations to the hyperbola and parabola are

$$a/x+b/y=1 \text{ and } \sqrt{x/a'}+\sqrt{y/b'}=1\,;$$

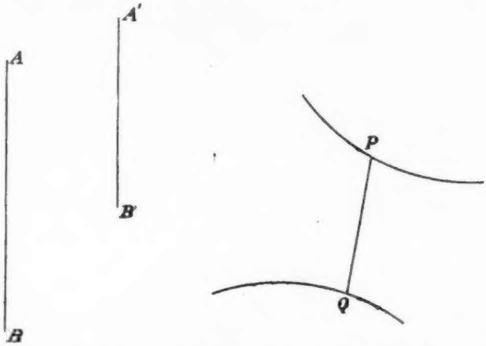
and if any point x', y' on the hyperbola be taken for the vertex of a circum-inscribed triangle, the equation to the opposite side of the triangle is

$$xx'/aa'+yy'/bb'=1.$$

We have thus a fairly simple method for investigating the properties of such triangles.

F. S. MACAULAY.

65. *Chords of quickest and slowest descent from one circle to another, both circles being in a vertical plane, and not intersecting.*



As is well known, if PQ be a chord of quickest or slowest descent from one curve to another in the same vertical plane, the verticals through P, Q and the normals at P, Q form a rhombus.

Further, it is easy to see that, if the centres of curvature at P and Q both lie outside the verticals, the time in PQ is a minimum, and if they both lie inside, the time in PQ is a maximum ; while the time is neither a maximum nor a minimum if neither of these conditions is fulfilled.

Let A, A' be the highest, B, B' the lowest points of the two circles. Join AA', AB', BA', BB'. Let P, Q be the points in which one of these chords cuts the circles again. Then the normals and verticals at P, Q form a rhombus, as may be seen at once by noting that PQ always passes through a centre of similitude.

By applying the criterion of the positions of the centres of curvature, we discriminate as follows :

(1) If the circles are external to each other, the chords through the centre of *inverse* similitude alone give solutions, and

(*a*) If B' is above A, the chord $A'B$ gives a minimum, the chord $B'A$ a maximum time of descent from the circle $A'B'$ to the circle AB.

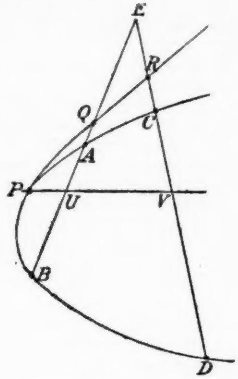
(*b*) If B' is below A, but A' above B, the same chords give two minimum solutions, one from the circle AB to the circle $A'B'$, one from the circle $A'B'$ to the circle AB.

(2) If one circle is inside the other, the chords through the centre of *direct* similitude alone give solutions, both minimum, one from the outer circle to the inner, one from the inner circle to the outer.

H. A. ROBERTS.

66. *Note on the parabola through four concyclic points.*

Let A, B, C, D be given fixed concyclic points upon a given parabola.



Produce AB, CD, which are equally inclined to the axis, to meet externally at E. At *any* point P on the curve draw the tangent PQR and the diameter PUV, meeting AB, CD in Q, R and U, V respectively.

Then, by a known property of the parabola,

$$QU^2 = QA \, . \, QB, \text{ and } RV^2 = RC \, . \, RD$$

(*Milne & Davis*, p. 26). Hence QR is the radical axis of the circle through A, B, C, D, and a circle touching EAB, ECD in U, V.

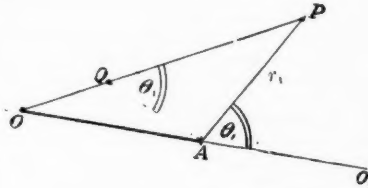
Conversely, the envelope of the radical axis of a given circle and a variable circle touching two given fixed straight lines is one of the two parabolas through the four points in which the straight lines intersect the fixed circle. [One parabola corresponds to the series of variable circles inscribed within the acute angle between the straight lines, and another to those within the obtuse angle.]

It will be seen that the above property furnishes a method of drawing the tangent *in any proposed direction* to the parabola through four given concyclic points; whence the tangent at the vertex and directrix can be determined.

(Cf. *Educational Times*, May, 1898, Question 13840.) R. F. Davis.

67. *Circles are drawn through two fixed points. Trace the path of a point which cuts them all at the same angle.* (Question 172, p. 89.)

Let O, A be the two fixed points.



Invert the system of circles, taking O as centre and OA $(=a)$ as radius of inversion. The inverted system is a system of straight lines through A.

If A is taken as pole, the spiral curve which cuts these straight lines at a given angle a has the equation

$$r_1 = be^{\mu\theta_1}, \text{ ....(1)}$$

where $\mu = \cot a$, and b is any constant length. The inverse of this spiral with regard to O is the curve required.

Let P be a point on the equiangular spiral, and let Q be its inverse point. Let

$$OQ = \rho, \qquad AQ = \rho',$$
$$QOA = \theta, \qquad QAO' = \theta'.$$

Then, in the similar triangles OAP, OQA,

$$\frac{r_1}{a} = \frac{\rho'}{\rho},$$

also
$$\theta' - \theta = \pi - \theta_1 ;$$

therefore from (1)

$$a\rho' = b\rho \, . \, e^{\mu(\pi+\theta-\theta')} = c \, . \, \rho e^{\mu(\theta-\theta')}, \text{ if } c = be^{\mu\pi}, \text{ i.e. } a\rho' e^{\mu\theta'} = c\rho e^{\mu\theta}.$$

This is the equation of the required curve in bipolar coordinates. Its equation in rectangular or polar coordinates could easily be deduced from this,

but would be too complicated for use in drawing the curve, which is probably most easily obtained from the intersections of the equiangular spirals

$$\rho = kae^{-\mu\theta}, \quad \rho' = kce^{-\mu\theta'},$$

drawn for different values of k, which is easily done after one has been drawn.

To save labour the values of k should proceed geometrically, and c should be equal to one of the values of ka. The two systems of curves are then merely differently placed repetitions of each other, one system having O as pole, the pole of the other system being A. Indeed, they are all repetitions of one equiangular spiral, its various positions round O (or A) being obtained by successive rotations round O (or A) through a definite angle each time if k increases geometrically as suggested.

The curves required pass through the intersections of consecutive spirals, and are themselves spiral curves, each coiling an infinite number of times round O in gradually widening circuits, and then coiling round A an infinite number of times in gradually decreasing circuits (or, of course, *vice versâ*). One of the curves (viz. when $c=a$) goes off to infinity after coiling round O, and comes back from the opposite direction of infinity to coil round A, its direction at infinity being parallel to the line $\theta=\alpha$.

If the coaxial circles have real limiting points instead of real points of intersection, O and A will be the limiting points, and μ will then $= \tan\alpha$ instead of $\cot\alpha$.

If A coincides with O, the curves become another similar system of circles through O, the common tangents at O to the two systems cutting each other at the required angle. A. LODGE.

[The general curve of the whole system for all values of the constant angle of section is a spiral with two poles O, A, which does not go to infinity, and which takes roughly the shape of the letter S. The whole system, however, includes three systems which form exceptions to this statement. One is the system which goes to infinity; and the other two are the system of circles through O, A, and a second system of circles cutting the first or given system orthogonally.

It is clear that any one of the spirals not only cuts the first system of circles at a constant angle, but also the second system. This second system is also a coaxial system, but with imaginary common points, having the poles of the spirals for limiting points. The spirals may be considered as being given by either system of circles.

The whole system of spirals when inverted with respect to any point becomes an exactly similar system, the new poles being the inverses of the old ones. This follows from the fact that angles are unaltered by inversion, and circles invert into circles.

If one pole goes to infinity, the spirals become equiangular spirals, with the remaining pole as pole. Hence, by regarding an equiangular spiral as having in reality two poles, one of which is at infinity, we may regard the system of spirals with two finite poles as being a very simple generalisation of equiangular spirals. From this aspect their properties with respect to inversion and their relation to infinity gain in interest.

A further generalization of an equiangular spiral is the curve cutting at a constant angle the system of circles which touch two given circles; and this includes, as a special case, the curve which cuts the tangents to a given circle at a constant angle. F. S. MACAULAY.]

68. *In a spherical quadrangle the arcs joining the middle points of the three pairs of opposite sides are concurrent.* (Question 90, p. 16. Taken from Casey's *Spherical Trigonometry.*)

To solve this problem by using the relations between the trigonometrical ratios of the arcs involved would appear to be a very laborious operation.

The fact of the problem being given in a work on trigonometry does not appear to render a geometrical solution inadmissible. This seems to be sufficiently indicated by the free use of purely geometrical methods in establishing some of the important propositions of spherical trigonometry. What, from its extreme simplicity, appears to me to be emphatically *the* solution of the problem, is as follows:

Let A, B, C, D be the four vertices of the quadrangle; L, M, N, L', M', N' the middle points of the arcs BC, CA, AB, AD, BD, CD; and l, m, n, l', m', n' the middle points of the corresponding chords. Then if O is the centre of the sphere, OlL is a straight line; similarly for OmM, $Ol'L'$, etc. Then the straight line lm is parallel to the chord AB and half of it, since l, m bisect the chords CB, CA. Similarly for $l'm'$. Thus $lml'm'$ is a parallelogram, and ll', mm' bisect one another. Thus ll', mm', nn' meet in a common point k; the planes Oll', Omm', Onn' (that is, the planes OLL', OMM', ONN') meet in a common line Ok; and the arcs LL', MM', NN' meet in a common point K, viz. the point where Ok cuts the surface of the sphere. J. C. PALMER.

EXAMINATION QUESTIONS AND PROBLEMS.

Our readers are earnestly asked to help in making this section of the GAZETTE *attractive by sending either original or selected problems.*

Solutions should be sent within three months of the date of publication. They should be written clearly on one side of the paper. Contractions not intended for printing should be avoided. Figures should be drawn with the greatest care on as small a scale as possible, and on a separate sheet.

The question need not be re-written, but the number should precede every solution.

The source of problems when not otherwise indicated is shown by —C. (Cambridge), O. (Oxford), D. (Dublin), W. (Woolwich), Sc. (Science and Art Department).

260. The tangent to an ellipse at P meets the equiconjugate diameters at T, T'. The tangent at P to the circle $T'CP$ meets CT in L. Show that $PL^2 = CL \cdot TL$. W. F. BEARD.

261. If opposite sides of a hexagon $ABCDEF$ be parallel, and another hexagon $PQRSTV$ be formed by intersections of AC and BD, BD and CE, CE and DF, etc., prove that Pascal's theorem holds true for all the hexagons (or hexagrams) formed by the points A, B, C, D, E, F, and that Brianchon's theorem holds for $PQRSTV$. W. S. COONEY.

262. P is a point on an ellipse, foci S, S'. PR, the symmedian through P of the triangle SPS', cuts the circumcircle in R. RO perpendicular to PR cuts the normal at P in O the centre of curvature at P. R. F. DAVIS.

263. $ABCD$ is a quadrilateral with given sides, on a fixed base AB. A triangle is inscribed in a fixed circle, sides parallel to BC, CD, DA respectively. Show that each side of the triangle envelopes a circle, and that the four circles are coaxial.

A. C. DIXON.

264. The circles in No. 248 cut on PQR. W. J. GREENSTREET.

265. If n be a composite number, what is its smallest value that satisfies the condition that $2^{n-1}-1$ should be divisible by n? G. HEPPEL.

266. In testing a number for possible divisors it suffices to test the primes up to its square root. E. HILL.

267. If the roots of the equation

$$a_0x^n - na_1x^{n-1} + \frac{n(n-1)}{1 \cdot 2}a_2x^{n-2} - \dots + (-1)^n a_n = 0$$

are all positive, show that $a_r a_{n-r} > a_0 a_n$ for all values of r between 1 and $n-1$ inclusive, unless the roots are all equal. A. LODGE.

268. The distance between the pair of inverse points common to two circles equals the product of the direct and transverse common tangents, real or imaginary, divided by the distance between the centres. C. E. McVICKER.

269. There are 15 boat clubs: 2 of the clubs have each 3 boats on the river, 5 others have 2, and the remaining 8 have 1; find an expression for the number of ways in which a list can be formed of the order of the 24 boats, observing that the second boat of a club cannot be above the first, or the third above the second. R. F. MUIRHEAD.

270. PQ are points of inverse coordinates $\left(i.e.\ h, k;\ \frac{a^2}{h}, \frac{b^2}{k}\right)$ with respect to the axes of an ellipse; F is the pole of the line PQ. Through F a straight line is drawn making with either axis an angle equal to that made by PQ but in the opposite direction. Prove that it bisects PQ.

Hence, given any diameter of the ellipse, find groups of four points on the circumference from each of which groups four normals drawn shall meet somewhere on that diameter. (See Problem 225.) E. P. ROUSE.

271. Find the number of ways of dividing a number n into not more than three parts. C. F. W. SANDBERG (O.).

272. Show that No. 214 and the generalization are particular cases of Pascal's Theorem. J. A. THIRD.

273. Two circles touch internally at P, and a chord AB of the outer touches the inner at C. If PA, PB cut the inner circle at D, E, show that $PD \cdot PE$ is to PC^2 as the radius of the inner circle is to the radius of the outer. J. V. THOMAS.

274. (Extension of 235.)

If $\sum_{r=1}^{r=n-1} x_r^2 - \operatorname{sech} a \sum_{r=1}^{r=n-2} x_r x_{r+1} = c$, prove that

(1) $x_r^2 \leq 2c\dfrac{\sinh(n-r)a \sinh ra \cosh a}{\sinh na \sinh a}$.

(2) $\dfrac{c\cosh a}{1+\cosh a} \leqslant \sum x^2_{n-1} \leqslant \dfrac{c\cosh a}{\cosh a - 1}$.

(3) The maximum and minimum values of $x_r x_{r+s}$ are

$$c\cosh a/(d_1 \sinh a) \quad \text{and} \quad c\cosh a/(d_2 \sinh a),$$

where d_1, d_2 are the roots of

$$(d \sinh sa + 1)^2 \sinh ra \sinh (n-r-s)a = \sinh (r+s)a \sinh (n-r)a.$$

A. C. L. WILKINSON.

275. Two light rods AB and AC, freely jointed at A, rest in a vertical plane; B and C are in contact with a smooth horizontal plane. Two other light rods DE and EF are *rigidly* connected at E, and hang with a heavy body supported at E, their ends D and F carrying smooth rings sliding on AB and AC respectively. Required the stress at A and the tensions in DE and EF. (C.)

276. Trace the curves

$$(a)\ y^3 - y^2 + 4x^2 - 27x^4 = 0, \qquad (b)\ a(x^2+y^2) = xy(x+y);$$

and show that the circle $x^2+y^2+2a(1+\sqrt{2})(x+y-2a)=2a^2$ has treble contact with the latter. (C.)

277. Show that the maximum value of the perpendicular from the pole of $r=a(1-\cos\theta)$ on the normal is $4\sqrt{3}\,a/9$. (C.)

278. The squares of two consecutive numbers are of the forms $11m+4$, $12n+1$; find the forms of the numbers. (C.)

279. m points are taken on each of n straight lines and all the points are joined by straight lines. Find the total number of intersections of lines in the figure. (C.)

SOLUTIONS.

A great number of solutions are in hand, and will be published as sufficient space is available.

Solutions are wanted for Nos. 171, 251, 252, 258.

Solutions of questions 172, 90 will be found among the Mathematical Notes, Nos. 67, 68.

219. *Eliminate θ and ϕ between*

$$ax \sec\theta - by \operatorname{cosec}\theta = ax \sec\phi - by \operatorname{cosec}\phi = c^2 = a^2 + b^2,$$

and

$$\cos\frac{\theta-\phi}{2} \sec\frac{\theta+\phi}{2} = \frac{c}{a}.$$

Solution by C. E. M^c^VICKER; F. A. FIELD.

Writing $ax=c\xi$, $by=c\eta$, $\tan\frac{1}{2}\theta=t_1$, $\tan\frac{1}{2}\phi=t_2$, first two equations become

$$\xi\sec\theta - \eta\operatorname{cosec}\theta = c = \xi\sec\phi - \eta\operatorname{cosec}\phi.$$

Now $\qquad \sec\theta=(1+t_1^2)/(1-t_1^2)$, $\operatorname{cosec}\theta=(1+t_1^2)/2t_1$;

thus t_1, t_2 are roots of $\xi(1+t^2)/(1-t^2)-\eta(1+t^2)/2t=c$,

i.e. of $\qquad \eta t^4 + 2(\xi+c)\,.\,t^3 + 2(\xi-c)\,.\,t - \eta = 0.$

Let t_3, t_4 be remaining roots, and putting

$$t_1+t_2=T,\ t_3+t_4=T',$$

we have
$$T+T'=-2(\xi+c)/\eta, \quad \text{.................................}(a)$$
$$-TT'=t_1t_2+t_3t_4, \quad \text{.................................}(b)$$
$$t_3t_4\,.\,T+t_1t_2\,.\,T'=-2(\xi-c)/\eta, \quad \text{.................................}(c)$$
$$t_1t_2\,.\,t_3t_4=-1.$$

But from third given equation $t_1t_2=(c-a)/(c+a)$,

and therefore $t_3t_4=-(c+a)/(c-a)$.

Using these in (a) and (c), we find

$$(c^2+a^2)T\eta=2(c-a)(a\xi-c^2),$$
$$(c^2+a^2)T'\eta=-2c(c+a)(\xi+a).$$

Eliminating T, T' by means of (b), and restoring x, y, there results

$$(c^2-a^2)(a^2x-c^3)(x+c)+(c^2+a^2)^2y^2=0,$$

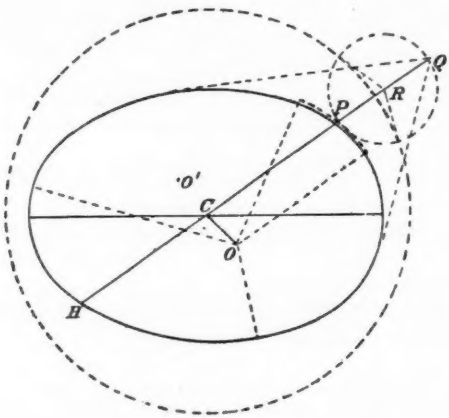
which, since $c^2=a^2+b^2$, may also be written,

$$a^2b^2(x+c)^2+(c^2+a^2)^2y^2=b^2c(c^2+a^2)(x+c).$$

[Cf. Salmon's *Conics*, § 226, Ex. 11.]

225. *P, Q are points of inverse coordinates $\left(i.e.\ h, k, \dfrac{a^2}{h}, \dfrac{b^2}{k}\right)$ with respect to the axes of an ellipse. Tangents are drawn to the ellipse from either, and the normals at the points of contact meet in O.*

Prove (a) that CO is perpendicular to PQ, and that $\dfrac{CO}{PQ}$ is the cotangent of the angle between CP and any chord which it bisects; (b) the director-circle and the circle on PQ as diameter are orthogonal. Hence show how to draw four normals from any given point in a diameter at right-angles to either of the equi-conjugates. E. P. Rouse.



Solution by Proposer and H. G. Mayo.

If P be (h, k), O is $\left[\dfrac{c^2h(b^2-k^2)}{b^2h^2+a^2k^2},\ -\dfrac{c^2k(a^2-h^2)}{b^2h^2+a^2k^2}\right]$ where $c^2=a^2-b^2$.

(*a*) (i.) The coordinates of O remain unchanged, save in sign, on substituting $\frac{a^2}{h}, \frac{b^2}{k}$ for h, k. Hence the pairs of normals corresponding to P and Q meet at points in a straight line with C and equidistant from it.

$$CO \text{ is } yh(b^2-k^2)=-xk(a^2-h^2).$$

$$PQ \text{ is } \frac{y-k}{x-h}=\frac{\frac{b^2}{k}-k}{\frac{a^2}{h}-h}=\frac{h}{k}\,\frac{b^2-k^2}{a^2-h^2}.$$

Thus CO is perpendicular to PQ.

(ii.) The angle between CP and a conjugate chord

$=$sum of acute angles made with the axis by CP and its conjugate

$=\tan^{-1}\frac{k}{h}+\tan^{-1}\frac{b^2h}{a^2k}$ (the latter is angle made by CQ)

$=\tan^{-1}\frac{a^2k^2+b^2h^2}{c^2hk}$.

By similar triangles $\frac{CO}{PQ}=\frac{\text{ordinate of } O}{\text{difference of abscissae of } P \text{ and } Q}=\frac{c^2hk}{a^2k^2+b^2k^2}$ whence (ii.) follows.

(*b*) If QM be perpendicular from Q to CP, square of tangent from C to circle PMQ

$$=CP\,.\,CM=CP\,.\,CQ\cos PCQ=h\,.\frac{a^2}{h}+k\,.\,\frac{b^2}{k}=a^2+b^2\,; \;\therefore\; \text{etc.}$$

If CO be perpendicular to one of the equi-conjugates CH (say), we may assume that P lies on that conjugate. If so, Q also lies on it, for if $\frac{k}{h}=\frac{b}{a}$, then $\frac{b^2}{k}\div\frac{a^2}{h}=\frac{b}{a}$. But we know the angle (θ) between the equi-conjugates, and as $CO=PQ\cot\theta$, PQ is known.

Draw the director-circle. By means of a concentric circle find on HC produced a point R from which a tangent to the director-circle shall be $\frac{1}{2}PQ$. A circle, centre R, radius $\frac{1}{2}PQ$, will cut CR in two points P and Q (say). From P and Q draw tangents to the ellipse if both are outside it. Normals at one pair of points of contact will meet in O, and normals at the other pair in O' where $O'C=CO$ and O' is in OC produced. Hence normals at two points on the curve diametrically opposite to this other pair will also meet in O. Thus four points on the curve are found, the normals at which meet in the given point.

Mr. Mayo proves this directly from the equations of the circle PQ, viz.

$$x^2+y^2-x(h^2+a^2)/h-y(k^2+b^2)/k+a^2+b^2=0,$$

and the director-circle $\quad x^2+y^2=a^2+b^2$.

For the last part he points out that, if O be any point on a diameter perpendicular to the equi-conjugate PH, we have

$$\frac{CO}{PH}=\frac{CO}{2CP}=\text{cot. of angle between } PQ \text{ and major axis}=\frac{a}{b}\,;$$

$$\therefore CP=\frac{b}{2a}CO.$$

Hence P (and similarly Q) is found.

Therefore, if O is given, find two points P and Q on the equi-conjugate, such that $CP=\frac{b}{2a}\,.\,CO=CQ$. The joins of O to the points of contact of tangents from P and Q to the ellipse are the four normals required.

226. *ABC is an acute-angled triangle, P its orthocentre. The product of the latera-recta of three circumscribing ellipses, having the sides respectively as minor axes, is AP . BP . CP. Discuss the corresponding case of ellipses and hyperbolas when triangle is obtuse.* G. HEPPEL.

Solution by H. G. MAYO.

Let KM be the semi-major of a circum-ellipse, of which AC is the minor axis, AC being bisected in M.

Let the latera-recta for BC, CA, AB be L_1, L_2, L_3. Draw BE perpendicular to AC, and let P be the orthocentre.

$$L_2=\frac{MA^2}{2KM}=\frac{b^2}{2KM}.\quad \text{Also}\quad \frac{BN^2 \text{ or } EM^2}{KM^2-(MN^2 \text{ or } BE^2)}=\frac{AM^2}{KM^2}=\frac{b^2}{4KM^2},$$

and substituting for EM, BE, we get

$$KM=bc\sin A/(2\sqrt{ac\cos A\cos C})\,;$$

$$\therefore\quad L_2=\frac{b}{c}\,\frac{\sqrt{ac\cos A\cos C}}{\sin A},$$

whence $L_1L_2L_3=abc\Pi\cot A=AP\,.\,BP\,.\,CP$.

An analytical solution was given by J. L. THOMAS.

228. *OP touches the circle APB, OAB is a secant through the centre. OL, OM are taken along OP equal to OA, OB respectively. PQ is perpendicular to AB; LX, MY, to OAB. Show LX = AQ, MY = BQ.* E. M. LANGLEY.

Solution by R. F. DAVIS; W. J. GREENSTREET.

$OL=OA$; $LX=$perpendicular from A on $OP=AQ$, for AP bisects $O\hat{P}Q$. Similarly $MY=BQ$.

Solution by W. S. COONEY.

Let C be centre.

$$XL=\frac{PQ\,.\,OL}{OP}=\frac{OL}{OC}\,.\,r=\frac{OA}{OC}\,.\,r=r-\frac{r^2}{OC}=r-CQ=AQ.$$

$$MY=\frac{PQ\,.\,OM}{OP}=\frac{OM}{OC}\,.\,r=\frac{OB}{OC}\,.\,r=r+\frac{r^2}{OC}=r+CQ=BQ.$$

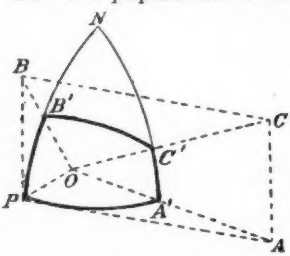
Solutions by E. LL. TANNER; J. L. THOMAS.

$\{OAQB\}$ is an H.R.; $\therefore$ PA, PB being at right angles bisect the angle (PQ, PO), *i.e.* $O\hat{P}A=A\hat{P}Q$.

$$\therefore\quad \left.\begin{matrix}OA:AQ\\ OB:BQ\end{matrix}\right\}::OP:PQ \quad\text{or}\quad ::OM:MY.\quad \text{But } OA=OL\,;\ OB=OM.$$

$$\therefore\quad AQ=LX\,;\ BQ=MY.$$

229. *OP is perpendicular to the plane of the rectangle APBC. The solid angle ω subtended by APBC at O is given by sin ω = sin α sin β, where α, β are the angles subtended at O by PA, PB.* A. LODGE.



Solution by C. E. McVICKER; H. P. KNAPTON.

With centre O, radius OP, draw a sphere; let OA, OB, OC pierce its surface at $A'B'C'$

Spherical excess of $A'PB'C'$

$=$(area of quad.)/(rad. of sphere)^2

$=$solid angle in question.

The planes OPA, OCA are orthogonal, for AC parallel to PB is normal to

pl. OPA. Hence A' and similarly B' are right angles;

$\therefore$ spherical excess $=$ excess of C' over a right angle.

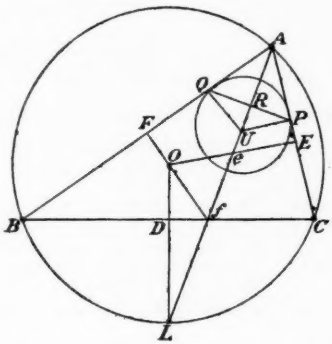
Let PB', $A'C'$ meet in N, the pole of PA'.

In the rt.-angled $\triangle\, NB'C'$, $\cos NC'B' = \sin N \cdot \cos NB'$;

$$\therefore \quad \sin\omega = \sin\alpha\sin\beta.$$

230. *Two points P, Q on the sides of any triangle are opposite corners of a rhombus, the others being the in-centre and the vertex. Show that the circle touching the sides at P and Q touches the circle through the ends of the base and the circum-centre.* C. E. M'VICKER.

Solution by R. F. DAVIS.



Let O be circum-centre of ABC and U be the centre of the circle touching CA, AB in P, Q respectively. DEF the median triangle of ABC. Produce OD to meet the circum-circle in L. Then AL bisects the angle A. Let OE, OF meet AL in e, f.

$$AP = AQ = AR\sec\frac{A}{2}$$

$$= \tfrac{1}{2}AI\sec\frac{A}{2} = \tfrac{1}{2}(s-a)\sec^2\frac{A}{2} = \frac{bc}{2s}.$$

We easily find that

$$ef = \tfrac{1}{2}(c-b)\sec\frac{A}{2};$$

$$\therefore \quad Oe = Of = \tfrac{1}{2}(c-b)\operatorname{cosec} A.$$

$$Ue \cdot Uf = PE \cdot QF\sec^2\frac{A}{2} = \left(\tfrac{1}{2}b - \frac{bc}{2s}\right)\left(\tfrac{1}{2}c - \frac{bc}{2s}\right)\sec^2\frac{A}{2} = bc(s-b)(s-c)\sec^2\frac{A}{2}\Big/(4s^2)$$

$$= b^2c^2\tan^2\frac{A}{2}\Big/(4s^2) = UP^2.$$

Therefore the circles, centres O, U, radii Oe, UP, are orthogonal.

The sides of OBC are as $2\sin A : 1 : 1$, and the tangents from O, B, C to the circle U are $\tfrac{1}{2}(c-b)\operatorname{cosec} A$, $c - \frac{bc}{2s}$, $b - \frac{bc}{2s}$; therefore by a well-known theorem the circle OBC touches the circle U.

[A solution by the PROPOSER will be found in the Notes, p. 190.]

231. *Two concentric and coaxial ellipses are so related that triangles circumscribed to the one are inscribed to the other. The normals to the former at the points of contact of the sides of any one of the triangles are concurrent, as also are the normals to the other ellipse at the angular points of the triangle.* G. RICHARDSON.

Solution by W. J. GREENSTREET.

Let $S \equiv \frac{x^2}{a^2} + \frac{y^2}{b^2} - 1 = 0$, $S' \equiv \frac{x^2}{a'^2} + \frac{y^2}{b'^2} = 1$ be two concentric and coaxial conics.

If α, β, γ be eccentric angles of three points on $S = 0$.

Eliminating a', b' from the conditions that the chords

$$\frac{x}{a}\cos\tfrac{1}{2}(\alpha+\beta) + \frac{y}{b}\sin\tfrac{1}{2}(\alpha+\beta) = \cos\tfrac{1}{2}(\alpha-\beta), \text{ etc.,}$$

are tangents to S', we get $\quad \Sigma\sin(\beta+\gamma) = 0$,

which shows that the normals to S at α, β, γ are concurrent.

Again, if the join of β and γ on S be a tangent at α' to S', we have

$$\cos\alpha' = \cos\frac{\beta+\gamma}{2}\Big/\cos\frac{\beta-\gamma}{2}\,;\quad \sin\alpha' = \sin\frac{\beta+\gamma}{2}\Big/\cos\frac{\beta-\gamma}{2},$$

and the normals at α', β', γ' are concurrent if $\Sigma\sin(\beta'+\gamma')$, *i.e.*

$$\Sigma\left(\sin\frac{\alpha+\gamma}{2}\cos\frac{\alpha+\beta}{2}+\cos\frac{\alpha+\gamma}{2}\sin\frac{\alpha+\beta}{2}\right)\Big/\left(\cos\frac{\gamma-\alpha}{2}\cos\frac{\alpha-\beta}{2}\right)=0,$$

and the L.H. is

$$\frac{1}{\Pi\cos\frac{\beta-\gamma}{2}}\Sigma\sin\frac{2\alpha+\beta+\gamma}{2}\cos\frac{\beta-\gamma}{2}$$

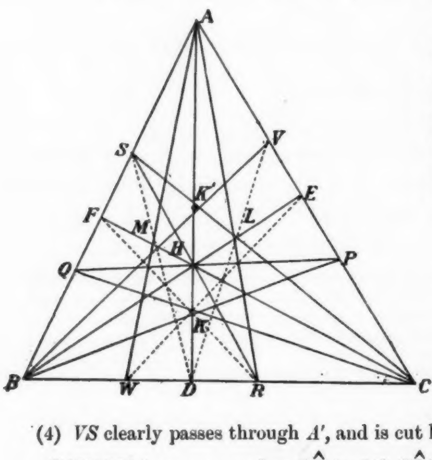
$$=\frac{1}{2\Pi\cos\frac{\beta-\gamma}{2}}\Sigma(\sin\overline{\alpha+\beta}+\sin\overline{\alpha+\gamma}),$$

which by the above is zero. Therefore, etc.

[For a proof that the locus of concurrence in each case is the same cubic, *v. E. Times Reprint*, Vol. LI., p. 51.]

232. *The polars of A, B, C with respect to semicircles on BC, CA, AB cut AB, AC in Q, P; BC, BA in R, S; CA, CB in V, W.*

(1) *The polars meet in the orthocentre H.*
(2) *BP, CQ meet in K on the perpendicular AD, and so on.*
(3) *E, K, W; F, K, R; etc., are collinear.*
(4) *LD, MD are equally inclined to AD, and so on.* R. TUCKER.



Solution by W. S. COONEY.

(1) If EF produced meet CB in A', $AA'H$ is a self-conjugate triangle to circle $BFEC$; $\therefore$ polar of A passes through H, etc. $(BQFA)$, $(CPEA)$ are harmonic sections, and similarly with $(CRDB)$, $(ASFB)$, $(AVEC)$, $(BWDC)$.

(2) $\therefore\ C(BQFA)=B(CPEA)$;

$\therefore\ A, H, K$ are collinear, etc.

(3) Also $R(ASFB)=W(AVEC)$;

$\therefore\ WE, RF$ intersect on AHD in the H.C. of A, *i.e.* in K; $\therefore$ etc. Also CS, BV intersect in K' on AD, etc.

(4) VS clearly passes through A', and is cut harmonically by AD, CB;

$\therefore\ D(VASB)$ is an H.P., and as $B\hat{D}A=90°$, $S\hat{D}V$ is bisected by AD; $\therefore$ etc.

[NOTE.—Let the semicircle on BC cut AB, AC in P', Q', and the polar PQ cut this circle in R, R'. Then the circle around APQ is the circle on AR' as diameter, R' being the orthocentre of ABC. And as AR is perpendicular to RR' (the polar PQ), it must pass through O, the mid-point of BC. Hence AR AR' are the median and perpendicular to BC from A.]

233. *A, B, and C each arrange* 105 *square tiles (side*=1 *decimetre) so that the outer and inner boundaries of the patterns are squares. Now the three patterns are all different and the outer boundary of A's pattern is as much longer than the inner boundary of B's pattern as the outer boundary of B's pattern is shorter than the inner boundary of C's pattern. Give the dimensions of A's pattern.* H. W. LLOYD TANNER.

Solution by E. LL. TANNER ; J. L. THOMAS ; H. P. KNAPTON.

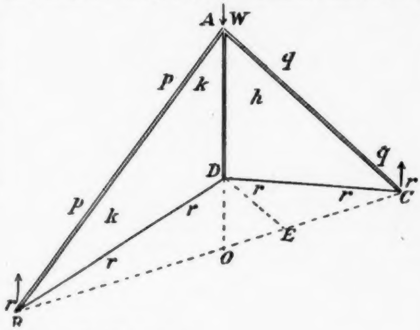
Let x, y, x_1, y_1, x_2, y_2 be the sides of outer and inner squares in the patterns of A, B, C.

Then $x^2-y^2=x_1^2-y_1^2=x_2^2-y_2^2=105$, leading to 11 and 4, 13 and 8, and 19 and 16 for the required dimensions of A, B, C respectively.

234. *Three fine light rods AB, AC, AD are freely jointed at A, and rest in a vertical plane on smooth horizontal supports at B and C under a load applied at A. The rod AD rests vertically downwards, and its extremity is connected with B and C by two fine light strings BD, CD. Prove that if T is the tension of the rod AD, and W the weight of the load at A, then*

$$T:W=OD:DA$$

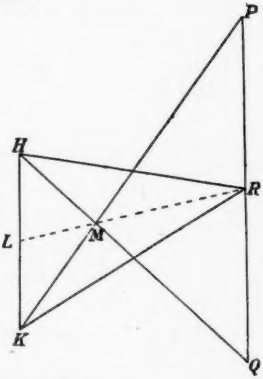
where O is the intersection of BC and AD produced. W. J. DOBBS.



Solution by PROPOSER.

Space diagram is marked with the letters p, q, r, h, k as in figure. The force diagram is P, Q, R, H, K, such that PQ represents W; QR and RP the reactions at C and B respectively; PK parallel to pk; RK to rk; QH to qh; RH to rh, and HK to hk.

Let RM meet HK in L, and draw DE parallel to AC to meet BC in E. The sides of $MKRHM$ and $ABDCA$ are parallel, as are the diagonals KH, AD. Hence MR is parallel to BC. (For a statical proof of this theorem in pure geometry, *v.* method in Dobbs, *El. Geom. Statics*, § 101.)



Now,

$$\frac{T}{W}=\frac{HK}{PQ}=\frac{LM}{MR}=\frac{OE}{EC}=\frac{OD}{DA}.$$

REVIEWS AND NOTICES.

Elementi di Geometria (2nd edition). By PROFESSORS G. LAZZERI and A. BASSANI. (Rafaello Giusti, Livorno.) The principal feature of this work, which, by omitting certain portions of the original 1891 edition, has been reduced in size, is its exposition of the elementary Geometry of Planes and Solids side by side. The great majority, if not all, of the theorems relating to plane figures have their analogues in theorems relating to space; and it is claimed that by

presenting these together the student obtains a clearer and more comprehensive view of geometry as a whole. This is no doubt true; but it may be objected that the ordinary student has not time to master all that is thus placed before him. To one who can give more than the usual amount of attention to mathematics the method adopted would prove advantageous.

The Elements of Euclid, rearranged in suitable divisions, form but a fraction of the work. The whole is divided into five books, each containing from four to seven chapters. Book I. deals with the segments of lines, plane and dihedral angles, first notions of the circle and sphere, parallel lines and planes, and perpendicular lines and planes. Book II. treats of polygons, solid angles, polyhedra, and distance, including under the last the properties of circumcentres, incentres, excentres, and centroids of triangles and tetrahedra. Book III. treats of the relations between lines, planes and spheres, of inscribed and circumscribed polygons and polyhedra, of geometry on the sphere, and of surfaces of rotation. Book IV. is on equivalence, that is, equality of magnitude apart from similarity of shape. This, as applied to the areas of plane figures and the superficial areas and volumes of solids, forms of itself a large subject. Book V. includes the theory of proportion, properties of similar figures, the measures of length, areas and volumes, the proofs of algebraic formulae for such measures, and ends with the numerical calculation of π.

The work comprises 380 closely printed pages, 312 diagrams, and 1067 exercises. The definitions stand out clearly from the text. The subject is treated very carefully and scientifically. The postulates seem, if anything, rather more numerous than necessary. Among the most interesting are the postulate of freedom of motion, the postulate of the existence of the straight line as that which remains fixed when any two points of a body are fixed and the body is rotated, the postulate of the existence of the plane, the postulate of return by forward rotation (stated only inferentially), the remarkable and important postulate of Archimedes, and the postulate of parallels in the form of Playfair's axiom. It is curious that, notwithstanding all that has been written on the subject, the question as to how far these postulates are independent of one another is little nearer to a settlement than it ever was. F. S. M.

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